

COMMENT

Comment on ‘Series expansions from the corner transfer matrix renormalization group method: the hard-squares model’

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Abstract. Earlier this year Chan extended the low-density series for the hard-squares partition function $\kappa(z)$ to 92 terms. Here we analyse this extended series focusing on the behaviour at the dominant singularity z_d which lies on the negative fugacity axis. We find that the series has a confluent singularity of order 2 at z_d with exponents $\theta = 0.83333(2)$ and $\theta' = 1.6676(3)$. We thus confirm that the exponent θ has the exact value $\frac{5}{6}$ as observed by Dhar.

In [1] Y. Chan extended the low-density series for the hard squares partition function $\kappa(z)$ to 92 terms. In a brief analysis of the associated magnetisation series $M(z) = \frac{d}{dz} \ln \kappa(z)$ Chan found that this series has a physical singularity at $z_c = 3.79635(9)$ with exponent $\alpha = 0.0020(17)$. This in complete agreement with more accurate numerical work that has unequivocally established that the critical behaviour of the hard-squares model is in the Ising universality class and hence $\alpha = 0$. The current best estimate for z_c is to our knowledge $z_c = 3.79625517391234(4)$ [2]. More interestingly Chan gives a very accurate estimate for the dominant singularity of $\kappa(z)$ which happens to lie on the negative real axis at $z_d = -0.119338818(6)$ with critical exponent $-\gamma = 0.171(14)$. Dhar [3] has related the hard-squares model to directed-site animals and analysed a 42-term hard-squares series for $\kappa(z)$ calculated by Baxter *et al* [4] and found that the critical exponent at z_d is $\theta = \frac{5}{6}$. Obviously, $\gamma = 1 - \theta = \frac{1}{6}$.

The results of Chan are thus quite surprising in that z_d is obtained to 10 digit accuracy and γ only to 2 digit accuracy. This immediately suggests that the critical behaviour at z_d is more complicated than assumed in Chan’s analysis. The most obvious complication is that the singularity at z_d contains confluent terms. In this comment we analyse the series for $\kappa(z)$ and demonstrate that this is indeed the case and we show that the critical point z_d is a double root. This refined analysis then allows us to obtain accurate estimates for the exponents at z_d , namely, $\theta = 0.83333(2)$ and $\theta' = 1.6676(3)$, which obviously confirms the observation by Dhar [3] that $\theta = \frac{5}{6}$.

To estimate the singularities and exponents of $\kappa(z)$ we (as did Chan) use the numerical method of differential approximants [5]. We refer the interested reader to

Table 1. Real zeroes of Q_3 and the associated exponents obtained from a homogeneous third order differential approximant with polynomials of degree 21.

Zero	Exponent	Zero	Exponent
3.797609243403	2.039756900951	-0.302260961239	3.431304831739
3.696982277961	7.327149084651	-0.379900722124	4.228776094690
-0.119338882393	0.748746994128	-0.741300568486	2.959881511581
-0.119338893414	2.752278507267	-0.985745094861	1.581123317769
-0.120041211934	2.016623831595	-1.980070905154	2.502245197257
-0.173738685995	8.477340013372	-3.832447177296	0.541758005446
-0.259388829117	1.066025488201		

[5] for details, and Chapter 8 of [6] for an overview of the method. Suffice to briefly say that a K 'th-order differential approximant to a function $F(z)$, for which one has derived a series expansion, is formed by determining the coefficients in the polynomials $Q_i(z)$ and $P(z)$ of order N_i and L , respective, so that the solution $\tilde{F}(z)$ to the inhomogeneous differential equation

$$\sum_{i=0}^K Q_i(z) \left(z \frac{d}{dz}\right)^i \tilde{F}(z) = P(z) \quad (1)$$

agrees with the series coefficients of $F(z)$. The possible singularities of the series appear as the zeros z_i of the polynomial $Q_K(z)$ and the associated critical exponents λ_i are obtained from the indicial equation. Note that not all roots of Q_K are actual singularities of the underlying series.

In table 1 we list all the real zeros of $Q_3(z)$ and the associated exponents as obtained from a homogeneous third order differential approximant with polynomials of degree 21. The exponents were calculated assuming that all the roots are distinct and hence of order 1. We immediately notice that if the two zeros close to z_d (bold-faced in the table) are distinct they lie incredibly close to one another. A more likely scenario is that the root at z_d is of order 2. If we assume this is the case and solve the resulting indicial equation (using the average of the two zeros for z_d) we get the exponent estimates 0.833329270 and 1.667679940, which immediately suggests that the leading exponent is $\theta = 5/6$ in agreement with Dhar's result and that possibly the sub-dominant exponent θ' is twice this. The zero at $\simeq -0.1200$ though very close to z_d is likely distinct from z_d .

In table 2 we list estimates for z_d and the two associated critical exponents obtained by averaging several second or third order inhomogeneous differential approximants with a given degree L of the inhomogeneous polynomial. The quoted error is simply the mean deviation of the approximants. We conclude that

$$z_d = -0.11933888(4), \quad \theta = 0.83333(2), \quad \theta' = 1.6676(3).$$

Clearly $\theta = \frac{5}{6}$ confirming Dhar's result. While θ' is tantalisingly close to twice θ our

Table 2. Estimates for the dominant singular point z_d and the associated exponents as obtained from second and third order differential approximants. L is the degree of the inhomogeneous polynomial. The entry for $L = 0$ are estimates from homogeneous approximants.

Second order approximants			
L	z_d	θ	θ'
0	−0.119338899(11)	0.8333312(12)	1.66864(62)
2	−0.11933888907(24)	0.83334442(22)	1.667787(20)
4	−0.1193388866(21)	0.8333408(38)	1.66760(16)
6	−0.1193388864(25)	0.8333403(81)	1.66753(23)
8	−0.1193388868(18)	0.8333425(35)	1.66758(15)
10	−0.11933888754(23)	0.83334442(22)	1.667643(20)
Third order approximants			
L	z_d	θ	θ'
0	−0.1193388890(18)	0.8333289(10)	1.66777(13)
2	−0.1193388859(24)	0.83320(30)	1.66745(25)
4	−0.1193388866(24)	0.83324(21)	1.66753(28)
6	−0.1193388849(33)	0.83318(24)	1.66735(38)
8	−0.1193388847(36)	0.83318(28)	1.66733(40)
10	−0.1193388837(16)	0.83348(14)	1.66718(19)

current best estimate does seem to exclude this possibility.

In conclusion we have analysed the series for $\kappa(z)$ using differential approximants and found that the dominant singularity at z_d appears to have order 2. When this is taken into account the method of differential approximants is perfectly well capable of yielding accurate exponent estimates. In particular we confirm that $\theta = \frac{5}{6}$ as found by Dhar [3].

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References

- [1] Chan Y 2012 Series expansions from the corner transfer matrix renormalization group method: the hard-squares model *J. Phys. A: Math. Theor.* **45** 0850013.
- [2] Guo W and Blöte H W 2002 Finite-size analysis of the hard-square lattice gas *Phys. Rev. E* **66** 046140.
- [3] Dhar D 1983 Exact solution of a directed-site animals-enumeration problem in three dimensions *Phys. Rev. Lett.* **51** 853
- [4] Baxter R J, Enting I G and Tsang S K 1980 Hard-squares lattice model *J. Stat. Phys.* **19** 461

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- [5] Guttmann A J 1989 Asymptotic analysis of power-series expansions in *Phase Transitions and Critical Phenomena* (eds. C Domb and J L Lebowitz) (New York: Academic) Vol. 13.
- [6] Guttmann A J, ed. 2009 *Polygons, Polyominoes and Polycubes* vol. 775 of *Lecture Notes in Physics* (Springer).